▶ MELISSA ANTONELLI, UGO DAL LAGO, DAVIDE DAVOLI, ISABEL OITAVEM AND PAOLO PISTONE, *Enumerating Error Bounded Polytime Algorithms Through Arithmetical Theories*.
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Since its early days, computer science has profoundly benefitted from the several interactions with mathematical logic. Among these, there is the possibility of characterizing fundamental complexity classes within a purely logical framework [6, 8], thus considering them from a new viewpoint, less dependent on concrete machine models and explicit resource bounds. Yet, classes defined on the bases of *randomized* algorithms [10] have remained difficult to capture with the tools of logic and this is especially true for semantic classes, as **BPP**, which are, by their nature, more challenging to enumerate via recursion-theoretic means. Currently, it is simply not known whether an effective enumeration of the aforementioned probabilistic classes is possible. In fact, the definition of **BPP**, being based on error-bounded machines, is not amenable to an enumeration, but nobody knows if alternative presentations of this same class support it. Indeed, the sparse contributions along these lines are usually themselves *semantic* [5, 9] or restricted to deal with syntactic randomized classes, like **PP** [3, 4].

The present work makes a step towards a logical characterization of randomized classes by considering a language in which the probability of error can be kept under control *from within* the logic. Concretely, we introduce a minimal extension of the language of arithmetic, such that the bounded formulas provably total in a suitably-defined theory *á la Buss* [2, 7] (expressed in this new language) precisely capture polytime *random* functions. Then, we provide two characterizations of **BPP**, obtained by internalizing the error-bound check *within* a logical system: one relies on measure-sensitive quantifiers [1], the other expresses the "measurement" via standard first-order quantification. This leads us to the introduction of a family of effectively-enumerable subclasses of **BPP**, each consisting of languages captured by probabilistic Turing machines whose underlying error can be proved bounded in the corresponding arithmetical theory. As a paradigmatic consequence, we establish that polynomial identity testing is in $\mathbf{BPP}_{\mathsf{PA}}$.

[1] Melissa Antonelli, Ugo Dal Lago, Paolo Pistone, *On Measure Quantifiers in First-Order Arithmetic*, ***Connecting with Computability*** (Ghent, July 5-9 2021), (Lisbeth De Mol, Andreas Weiermann, Florin Manea, and David Fernández-Duque, editors), Springer, 2021, pp. 12–24.

[2] Samuel Buss, ***Bounded Arithmetic***, Princeton University, 1986.

[3] Ugo Dal Lago, Reinhard Kahle, and Isabel Oitavem, *A Recursion-Theoretic Characterization of the Probabilistic Class PP*, ***46th International Symposium on Mathematical Foundations of Computer Science (MFCS 2021)*** (Filippo Bonchi, and Simon J. Puglisi, editors), vol. 202, Leibniz International Proceedings in Informatics (LIPIcs), 2021, pp. 1–12.

[4] Ugo Dal Lago, Reinhard Kahle, and Isabel Oitavem, *Implicit Recursion-Theoretic Characterization of Counting Classes*, ***Archive for Mathematical Logic***, vol. 61 (2022), pp. 1129–1144.

[5] Ugo Dal Lago, and Paolo Parisen Toldin, *A Higher-Order Characterization of Probabilistic Polynomial Time*, ***Information and Computation***, vol. 241 (2015), pp. 114–141.

[6] Roland Fagin, *Generalized First-Order Spectra and Poynomial-Time Recognizable Sets*, ***Complexity of Computation*** (Richard Karp, editor), vol. 7, (1974), SIAM-AMS Proceedings, pp. 43–273.

[7] Fernando Ferreira, *Polynomial-Time Computable Arithmetic*, ***Logic and Computation*** (Wilfried Sieg, editors), vol. 106, AMS, 1990, pp. 137–158.

[8] Neil, Immerman, ***Descriptive Complexity***, Springer, 1999.

[9] Emil Jerábek, *Dual Weak Pigeonhole Principle, Boolean Complexity, and Derandomization*, ***Annals of Pure and Applied Logic***, vol. 129 (2004), no. 1, pp. 1–37.

[10] Rajeev Motwani and Prabhakar Raghavan, ***Randomized Algorithms***, Cambridge University Press, 1995.