▸ DOMENICO CANTONE, PIETRO CAMPOCHIARO, LUCA CUZZIOL, AND EUGENIO G. OMODEO, *Recursively enumerable sets and Diophantine finite-fold-ness*.
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Can we extract, from an available proof that each

$$D_a(x_1, \ldots, x_k) \quad = \quad 0\,, \qquad\qquad a \in \mathbb{N}\,,$$

in some indexed family of equations has at most one solution in $\mathbb{N}$, an effective bound $\mathscr{C}_a$ such that when $x_1 = \boldsymbol{v}_1 \quad , \ldots , \quad x_k = \boldsymbol{v}_k$ solves $D_a = 0$ then $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k \leqslant \mathscr{C}_a$? In 1974 (cf. [3]), Yuri V. Matiyasevich provided a negative answer referring to a family of Diophantine equations that involve exponentiation, $u \mapsto 2^u$, and speculated that an alike limitation holds for some collection of polynomials D_a with integral coefficients.

The said limiting result relies, in part, on the fact that the graph

$$\mathcal{F}(a\,,\,b) \quad \Longleftrightarrow \quad F(a) = b$$

of any primitive recursive function $F : \mathbb{N} \longrightarrow \mathbb{N}$ can be specified in the form

$$\exists\, x_1 \cdots \exists\, x_k \quad \varphi(\overbrace{a, b}^{\text{parameters}} \;,\; \overbrace{x_1, \ldots, x_k}^{\text{unknowns}})\,,$$

where φ is an arithmetic formula not involving universal quantifiers, negation, or implication. *Representability* in this form can be achieved even if solely addition and multiplication operators (along with equality and with positive integers) are adopted as primitive symbols of the arithmetic signature; but can representability be reconciled with *univocity*, to wit,

$$\exists\, x_1 \cdots \exists\, x_k \,\forall\, y_1 \cdots \forall\, y_k \;\Big[\, \varphi(a, b, y_1, \ldots, y_k) \implies \&_{i=1}^{k} (y_i = x_i) \Big]\,,$$

or (at worst) with *finite-fold-ness*

$$\exists\, x \,\forall\, y_1 \cdots \forall\, y_k \;\Big[\, \varphi(a, b, y_1, \ldots, y_k) \implies \textstyle\sum_{i=1}^{k} y_i \leqslant x \Big]\,,$$

without calling into play one extra operator designating $u \mapsto 2^u$?

As a preparatory measure towards a hoped-for positive answer to this question, one may consider surrogating the exponentiation operator by a relator $\mathscr{J}(u, v)$ designating an exponential-growth relation (a notion made explicit by Julia Bowman Robinson in 1952). To meet our desiderata, such a relation should be representable in polynomial terms and should link with each u in its domain only a finite number of v's. A promising recipe for constructing such a relation, advanced by Martin Davis in 1968, has been recently reused to construct five new candidate relations. Unfortunately, establishing whether a potential candidate is apt to the job calls for the hard task of proving that at least one of a few special quaternary quartic equations, each corresponding to one of the Heegner numbers $2, 3, 7, 11, 19, 43, 67, 163$, has a finite overall number of integral solutions.

The following synoptic table shows the candidate 'rule-them-all' equations obtained (cf. [1]) through the construction pattern proposed in Davis' 1968 paper [2]. Each such equation is associated with one of the nine so-called Heegner numbers; today we know that proving that any of the quartics

$$\begin{array}{r||c} d = 2 & 2 \cdot \left(r^2 + 2\,s^2\right)^2 - \left(u^2 + 2\,v^2\right)^2 \;=\; 1 \\ 3 & 3 \cdot \left(r^2 + 3\,s^2\right)^2 - \left(u^2 + 3\,v^2\right)^2 \;=\; 2 \\ 7 & 7 \cdot \left(r^2 + 7\,s^2\right)^2 - 3^2 \cdot \left(u^2 + 7\,v^2\right)^2 \;=\; -2 \\ 11 & 11 \cdot \left(r^2 + r\,s + 3\,s^2\right)^2 - \left(v^2 + v\,u + 3\,u^2\right)^2 \;=\; 2 \\ 19 & 19 \cdot 3^2 \cdot \left(r^2 + r\,s + 5\,s^2\right)^2 - 13^2 \cdot \left(v^2 + v\,u + 5\,u^2\right)^2 \;=\; 2 \\ 43 & 43 \cdot \left(r^2 + r\,s + 11\,s^2\right)^2 - \left(v^2 + v\,u + 11\,u^2\right)^2 \;=\; 2\ , \end{array}$$

associated with the respective Pell equations $x^2 - d\,y^2 = 1$ has only a finite number of solutions in $\mathbb{Z}$ would suffice to ensure that every recursively enumerable set admits a finite-fold polynomial Diophantine representation.

If the equation associated with d is finite-fold, then the following dyadic relation $\mathscr{M}_d$ over $\mathbb{N}$ admits a polynomial Diophantine representation:

$$d \in \{2, 7\}: \quad \mathscr{M}_d(p, q) \;:=\; \exists\, \ell > 4 \left[q = \boldsymbol{y}_{2^\ell}(d) \;\;\&\;\; p \mid q \;\;\&\;\; p \geqslant 2^{\ell+1} \right],$$

$$d \in \{3, 11, 19, 43\}: \quad \mathscr{M}_d(p, q) \;:=\; \exists\, \ell > 5 \left[q = \boldsymbol{y}_{2^{2\,\ell+1}}(d) \;\;\&\;\; p \mid q \;\;\&\;\; p \geqslant 2^{2\,\ell+2} \right],$$

where $\langle \boldsymbol{y}_i(d) \rangle_{i \in \mathbb{N}}$ is the endless, strictly ascending, sequence consisting of all solutions in $\mathbb{N}$ to the said equation $d\,y^2 + 1 = \square$. Independently of representability, each $\mathscr{M}_d$ turns out to satisfy J. Robinson's exponential growth criteria and Y. Matiyasevich's condition (cf. [5]):

> Integers $\alpha > 1$, $\beta \geqslant 0$, $\gamma \geqslant 0$, $\delta > 0$ exist such that to each $w \in \mathbb{N} \setminus \{0\}$ there correspond u, v such that: $\mathscr{M}(u\,,\,v)$, $u < \gamma\,w^\beta$, and $v > \delta\,\alpha^w$ hold.

It is very hard to guess whether the number of solutions to any of the six quartics shown above is finite or infinite. For quite a while the authors hoped that Matiyasevich's surmise that each r.e. set admits a single-fold polynomial Diophantine representation could be established by just proving that the sole solution to the quartic $2 \cdot \left(r^2 + 2\,s^2\right)^2 - \left(u^2 + 2\,v^2\right)^2 \;=\; 1$ in $\mathbb{N}$ is $\langle \bar{r}, \bar{s}, \bar{u}, \bar{v} \rangle = \langle 1, 0, 1, 0 \rangle$; but Evan O'Dorney (University of Notre Dame) and Bogdan Grechuk (University of Leicester) sent us kind communications that they had found two, respectively three, non-trivial solutions to this equation. The least solution is:

$$\begin{array}{rcl} r_1 &=& 87785870585342068062926200081436608184268655514367, \\ s_1 &=& 17971393248825651975481341050901530371301499434440, \\ u_1 &=& 5221618295817678692343699483662704959631052331713, \\ v_1 &=& 6739958317343073985310999451965479560858521871624; \end{array}$$

the components of the third solution are numbers of roughly 180 decimal digits each.

[1] D. Cantone, L. Cuzziol, and E. G. Omodeo, *Six equations in search of a finite-fold-ness proof*, `arXiv:2303.02208` (2023)

[2] Martin Davis, *One equation to rule them all*, ***Transactions of The New York Academy of Sciences. Series II***, vol. 30 (1968), no. 6, pp. 766–773.

[3] Yuri V. Matiyasevich, *Sushchestvovanie neèffektiviziruemykh otsenok v teorii èkponentsial'no diofantovykh uravneniĭ*, ***Zapiski Nauchnykh Seminarov Leningradskogo Otdeleniya Matematicheskogo Instituta im. V. A. Steklova AN SSSR (LOMI)***, vol. 40 (1974), pp. 77–93. (Russian. Translated into English as [4])

[4] ——— *Existence of noneffectivizable estimates in the theory of exponential Diophantine equations*, vol. 8 (1977), no. 3, pp. 299–311. (Translated from [3])

[5] ——— *Towards finite-fold Diophantine representations*, ***Journal of Mathematical Sciences***, vol. 171 (2010), no. 6, pp. 745–752.