▸ CIPRIANO JUNIOR CIOFFO, JACOPO EMMENEGGER, FABIO PASQUALI, GIUSEPPE ROSOLINI, *Grothendieck topologies and weak limits for constructive mathematics.*
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The construction of the elementary quotient completion of an elementary doctrine is an excellent tool to produce models of constructive theories for mathematics, see [3, 4]. The construction freely adds quotients of (definable) equivalence relations to an elementary doctrine, which is an algebraic description of a logical theory with equality.

The elementary quotient completion extends the well-known categorical construction of the exact completion $\mathcal{A}_{\mathrm{ex/wlex}}$ of a given category $\mathcal{A}$ with weak limits [1] provided that products are strong. In fact, most of the works that characterise models as exact completions invoke that the given category $\mathcal{A}$ has strong finite products. This matches with the situation of an elementary doctrine, whose base category is required to have finite products. Indeed, equality involves considering pairs of elements.

On the other hand, the peculiarity of strong finite products with respect to weak limits is certainly apparent. In the work [2] for his PhD thesis, one of the collaborators determined suitable conditions to present an extension of the notion of elementary doctrine with respect to a base category $\mathcal{B}$ with just weak finite products. It requires that equality behaves with some kind of bias with respect to a specific weak product diagram—hence the name ***biased elementary doctrine***. He also showed how the elementary quotient completion extends to the wider settings as a 2-functorial left adjoint.

We show how the two extensions refer to the same situation which involves the product completion $\mathcal{A}_{\mathrm{pr}} := (\mathrm{Fam}_{\mathrm{fin}}(\mathcal{A}^{\mathrm{op}}))^{\mathrm{op}}$ of a category $\mathcal{A}$. When $\mathcal{A}$ has weak finite limits there is a Grothendieck topology Θ where covers contain a diagram of weak binary products. This observation allows us to state our main results.

THEOREM 1. *Let $\mathcal{A}$ be a category with weak limits. Let P be a doctrine on $\mathcal{A}_{\mathrm{pr}}$ which is a Θ-sheaf. The following are equivalent:*
(i) *The doctrine P is elementary.*
(ii) *The restriction of P to $\mathcal{A}$ is biased.*

THEOREM 2. *Let $\mathcal{A}$ be a category with weak limits. There is a full embedding of categories $\mathcal{A}_{\mathrm{ex/wlex}} \hookrightarrow \mathrm{sh}(\mathcal{A}_{\mathrm{pr}}, \Theta)$ of the exact completion into the topos of Θ-sheaves. The embedding is exact and preserves any local exponential which exists in $\mathcal{A}_{\mathrm{ex/wlex}}$.*

[1] A. CARBONI AND E.M. VITALE, *Regular and exact completions*, ***Journal of Pure and Applied Algebra***, vol. 125 (1998), pp. 79–117.
[2] C.J. CIOFFO, ***Homotopy setoids and generalized quotient completion***, PhD thesis, Università degli studi di Milano, 2022.
[3] M.E. MAIETTI AND G. ROSOLINI, *Elementary quotient completion*, ***Theory and Applications of Categories***, vol. 27 (2013), pp. 445–463.
[4] ——— *Quotient completion for the foundation of constructive mathematics*, ***Logica Universalis***, vol. 7 (2013), no. 3, pp. 371–402.