Large cardinals and the Ultrapower Axiom

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Outline

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- Inner models and large cardinals: their virtues and limitations with respect to the independence phenomena in set theory

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- The Ultrapower Axiom and the inner model problem
- The Ultrapower Axiom in nature: from large cardinal axioms beyond choice to determinacy principles

Inner models and large cardinals

The Ultrapower Axiom

The Ultrapower Axiom in nature

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- Seeking to remedy this, Gödel suggested two directions: inner models and large cardinals

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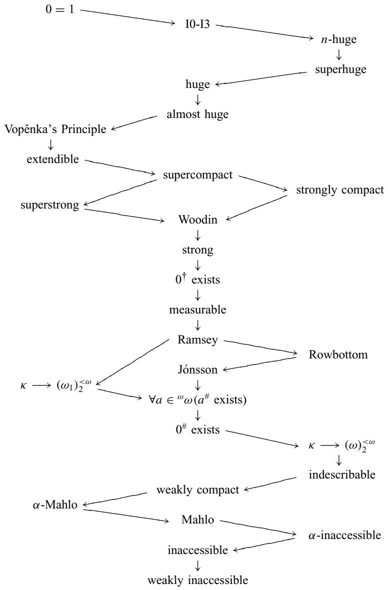
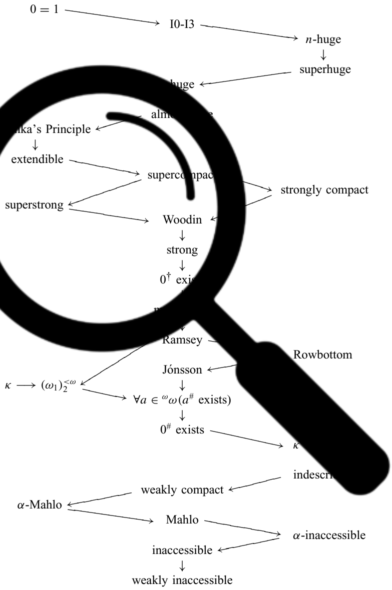
The proposition A added as a new axiom seems to give a natural completion of the axioms of set theory, in so far as it determines the vague notion of an arbitrary infinite set in a definite way. In this connection it is important that the consistency-proof for A does not break down if stronger axioms of infinity (e.g., the existence of inaccessible numbers) are adjoined to T. Hence the consistency of A seems to be absolute in some sense, although it is not possible in the present state of affairs to give a precise meaning to this phrase.

Large cardinal axioms

Large cardinal axioms

For first of all the axioms of set theory by no means form a system closed in itself, but, quite on the contrary, the very concept of set[17] on which they are based suggests their extension by new axioms which assert the existence of still further iterations of the operation "set of." These axioms can also be formulated as propositions asserting the existence of very great cardinal numbers or (which is the same) of sets having these cardinal numbers. The simplest of these strong "axioms of infinity" assert the existence of inaccessible numbers (and of numbers inaccessible in the stronger sense) $>\aleph_0$. The latter axiom, roughly speaking, means nothing else but that the totality of sets obtainable by exclusive use of the processes of formation of sets expressed in the other axioms forms again a set (and, therefore, a new basis for a further application of these processes).


0 = 1
I0-I3
n-huge
superhuge
huge
almost huge
Vopěnka's Principle
extendible
supercompact
strongly compact
superstrong
Woodin
strong
$0^\dagger$ exists
measurable
Ramsey
Rowbottom
Jónsson
$\kappa \longrightarrow (\omega_1)_2^{<\omega}$
$\forall a \in {}^\omega\omega (a^\# \text{ exists})$
$0^\#$ exists
$\kappa \longrightarrow (\omega)_2^{<\omega}$
indescribable
weakly compact
α-Mahlo
Mahlo
α-inaccessible
inaccessible
weakly inaccessible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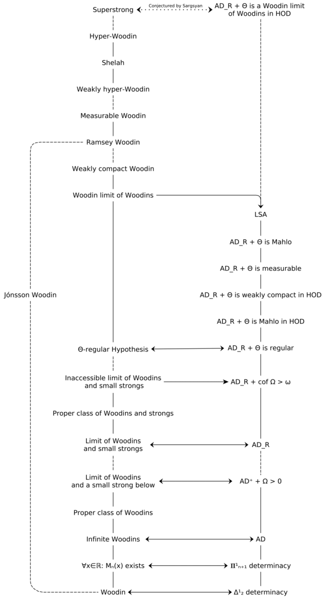
Superstrong
Conjectured by Sargsyan
AD_R + Θ is a Woodin limit of Woodins in HOD
Hyper-Woodin
Shelah
Weakly hyper-Woodin
Measurable Woodin
Ramsey Woodin
Weakly compact Woodin
Woodin limit of Woodins
LSA
AD_R + Θ is Mahlo
AD_R + Θ is measurable
Jónsson Woodin
AD_R + Θ is weakly compact in HOD
AD_R + Θ is Mahlo in HOD
Θ-regular Hypothesis
AD_R + Θ is regular
Inaccessible limit of Woodins and small strongs
AD_R + cof Ω > ω
Proper class of Woodins and strongs
Limit of Woodins and small strongs
AD_R
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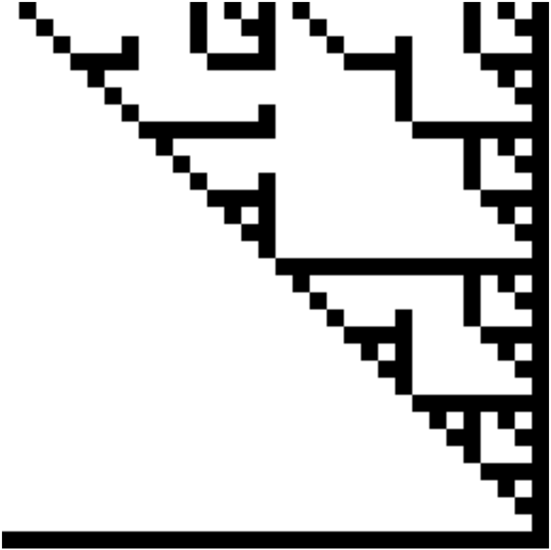
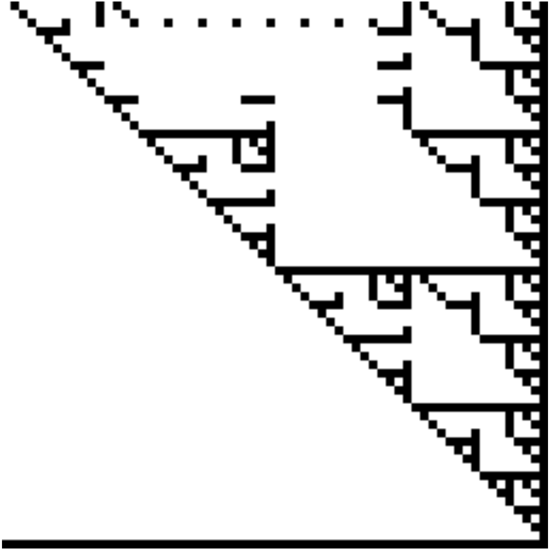
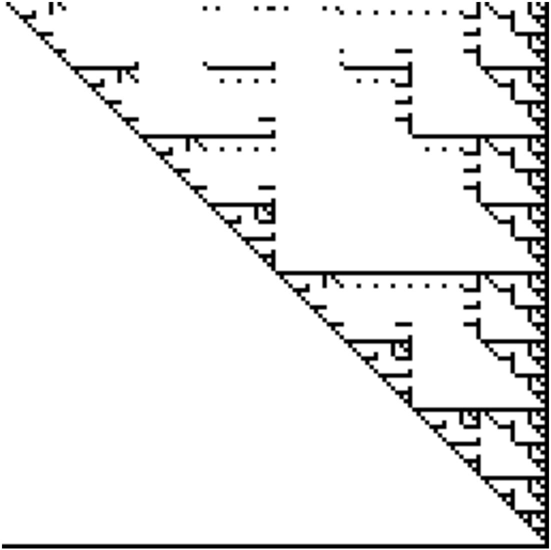
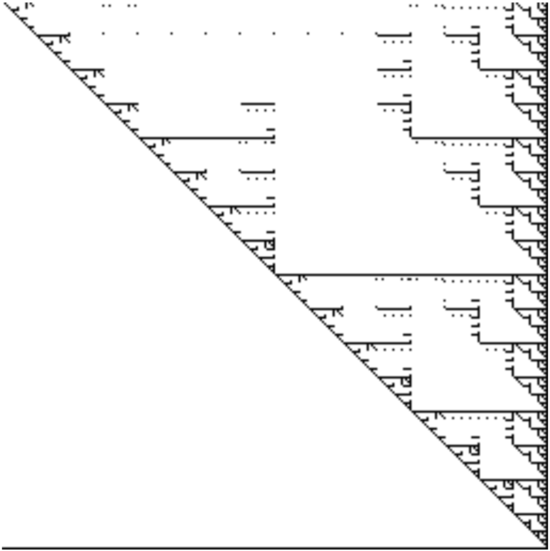
$AD^+ + \Omega > 0$
Proper class of Woodins
Infinite Woodins
AD
$\forall x \in \mathbb{R}$: $M_n(x)$ exists
Π^1_{n+1} determinacy
Woodin
Δ^1_2 determinacy









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given by $j_U(a) = [c_a]_U$ where $c_a : X \to \{a\}$ is the constant function

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Theorem (Scott)

If there is a measurable cardinal, the Axiom of Constructibility is false; that is, there is a nonconstructible set.

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- He also classified κ-complete ultrafilters in $L[U]$ and elementary embeddings from $L[U]$ to an inner model of $L[U]$

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Roughly speaking this asserts that any canonical inner models M and N have iterated ultrapowers M^* and N^* such that either $M^* \in N^*$, $N^* \in M^*$, or $N^* = M^*$

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Is there a canonical model with a strongly compact cardinal? Is there a comparison lemma at the level of strong compactness?

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This enables us to evaluate the prospects for the inner model theory of strongly compact cardinals: optimistically, to guide the way there; pessimistically, to prove an anti-inner model theorem

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- $N = \mathrm{Ult}(M, U)$
- $j = j_U$

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- UA is not provable from ZFC or from large cardinal axioms
- Instead, UA is a structure principle for large cardinals

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UA *holds iff for all* $\delta \in \mathrm{Ord}$, *the Ketonen order on* $\beta_{\aleph_1}(\delta)$ *is linear.*

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Theorem (G.)

Under the Ultrapower Axiom, the Mitchell order well orders the normal ultrafilters on any measurable cardinal.

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Remark: UA does not settle the Continuum Hypothesis itself

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Theorem (G.)

Under the Ultrapower Axiom, there is a subset of the least strongly compact cardinal from which every set is ordinal definable. As a consequence, V is a forcing extension of HOD.

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The proof uses the Axiom of Choice in an essential way. We now turn to large cardinals at this level in the *absence* of choice

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Theorem (G.-Schlutzenberg)

If $j : V_\alpha \to V_\alpha$ is elementary, then j is definable over V_α if and only if α is odd.

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On the other hand, assuming the Axiom of Determinacy, Θ is a strong limit cardinal — i.e., for all $\eta < \Theta$, $P(\eta)$ does not surject onto Θ — while $\aleph^*(V_{\omega+2}) = \Theta^+$

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Assuming AC, $\aleph^*(X) = |X|^+$, so e.g., $\Theta = (2^{\aleph_0})^+$ and GCH holds iff $\aleph^*(V_{\omega+\alpha}) = \aleph_{\alpha+1}$ for all $\alpha \in \text{Ord}$

On the other hand, assuming the Axiom of Determinacy, Θ is a strong limit cardinal — i.e., for all $\eta < \Theta$, $P(\eta)$ does not surject onto Θ — while $\aleph^*(V_{\omega+2}) = \Theta^+$

Theorem (G.)

Suppose there is a Reinhardt cardinal. Then for all sufficiently large ordinals α, $\aleph^*(V_\alpha)$ is a strong limit cardinal if and only if α is odd.

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Theorem (G.)

If there is a Reinhardt cardinal, then for sufficiently large γ, for all δ, $\beta_\gamma(\delta)$ is "almost" linearly ordered by the Ketonen order: any set of incomparables in $\beta_\gamma(\delta)$ has cardinality less than γ.

Inner models and descriptive set theory

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What is $\text{HOD}^{L(\mathbb{R})}$? It turns out to be a *canonical* inner model

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Sargsyan proved this up to the least Γ such that $L(\Gamma) \vDash \mathrm{AD}_{\mathbb{R}} + \Theta$ is regular by analyzing $L(\Gamma)$ as a canonical inner model. The hope is that $\mathrm{HOD}^{L(\Gamma)}$ is always a canonical inner model

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Larger pointclasses yield larger inner models. It's hoped that by considering large enough Γ, one obtains arbitrarily strong large cardinal axioms in the canonical inner model $\mathrm{HOD}^{L(\Gamma)}$

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The proof of Woodin's theorem shows that $\mathrm{HOD}^{L(\Gamma)}$ satisfies strong generalizations of UA, opening up a whole new line of research into comparison principles. For example:

Theorem (G.)

Assume $L(\Gamma) \vDash \mathrm{AD}^+$, and work in $\mathrm{HOD}^{L(\Gamma)}$. If δ is not a limit of cardinals κ such that κ is κ^+-supercompact, then the Mitchell order well orders the $(\delta, 2^{2^\delta})$-extenders E with $V_{\delta+2} \subseteq \mathrm{Ult}(V, E)$.

An approach to the consistency of UA

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The axiom I_0 states that there is an elementary embedding $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ with $\text{crit}(j) < \lambda$. This is one of the strongest large cardinal axioms compatible with the Axiom of Choice

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Assuming I_0 holds at λ, Cramer and Woodin showed that after collapsing λ to become countable, there is a pointclass Γ such that $L(\Gamma) \vDash \text{AD}^+$ and $\Theta^{L(\Gamma)} = (\aleph^*(V_{\lambda+1}))^{L(V_{\lambda+1})}$

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Possibility: $\text{HOD}^{L(\Gamma)} \vDash$ there is a huge cardinal

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- Prospects...

Thanks