Definable refinements of classical algebraic invariants

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Topogy and Classification

In topology one tries to classify spaces up to homeomorphism

In homotopy theory the relation of homotopy equivalence is considered

Invariants in Algebraic Topology

One attaches to topological spaces algebraic invariants such as groups

(All the groups will be abelian.)

From complexes to groups

The final invariant (group) is obtained by passing via complexes.

Why Polish groups?

Polish: second countable, topology induced by a complete metric

The class of Polish groups:

- contains locally compact groups
- contains spaces from analysis (Banach spaces, operator algebras)
- contains automorphism groups of “reasonable” structures
- is closed under countable products and inverse limits
- is closed under closed subgroups and quotients by closed subgroups
- the σ-algebra of Borel sets of a Polish group is standard (isomorphic to the σ-algebra of Borel sets of $\mathbb{R}$)

The homology of a Polish complex

Consider a complex of Polish groups A_*:

$$\cdots \longrightarrow A_2 \xrightarrow{\varphi_2} A_1 \xrightarrow{\varphi_1} A_0 \longrightarrow \cdots$$

In A History of Algebraic and Differential Topology, Dieudonné writes of

> *a trend that was very popular until around 1950 (although later all but abandoned), namely, to consider homology groups as topological groups for suitably chosen topologies.*

In 1976 Calvin C. Moore writes about

> *one final difficulty in considering the cohomology of topological groups which to some extent is incurable, and this is the fact that a continuous group homomorphism need not have closed range.*

The problem with cokernels

Problem: the category $\mathcal{A}$ of abelian Polish groups is not abelian

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The derived category of abelian Polish groups and its heart

Let $\mathcal{A}$ be the category of abelian Polish groups

Let $\mathrm{K}(\mathcal{A})$ be the triangulated category of complexes over $\mathcal{A}$

The derived category $\mathrm{D}(\mathcal{A})$ of $\mathcal{A}$ is obtained from $\mathrm{K}(\mathcal{A})$ by formally inverting the morphisms whose mapping cone is acyclic

The heart (or coeur)

$$\mathrm{LH}(\mathcal{A}) \subseteq \mathrm{D}(\mathcal{A})$$

is defined with respect to the canonical (left) truncation-structure on $\mathrm{D}(\mathcal{A})$

- $\mathrm{LH}(\mathcal{A})$ is the "smallest" abelian category containing $\mathcal{A}$
- $\mathrm{D}(\mathcal{A}) = \mathrm{D}(\mathrm{LH}(\mathcal{A}))$

More generally the same applies to any quasi-abelian category

An explicit description of the heart of abelian Polish groups

Theorem (L., 2022)

Explicit description of $\mathrm{LH}(\mathcal{A})$ *as a concrete category*

Objects: *"Formal quotients"* $\hat{G}/N$ *of abelian Polish groups by Polish subgroups (the topology of* N *need not be induced by* $\hat{G}$*)*

Morphisms: *Group homomorphisms* $\hat{G}/N \to \hat{H}/M$ *that are Borel-definable, i.e. induced by a Borel function* $\hat{G} \to \hat{H}$

Techniques: advanced tools and recent results from logic

$\mathrm{LH}(\mathcal{A})$ is the natural framework to develop definable refinements of classical homological algebra and algebraic topology

An explicit description of the heart of other categories

Similar descriptions for the heart of other topological-algebraic structures:

- locally compact abelian Polish groups
- totally disconnected locally compact abelian Polish groups
- non-Archimedean abelian Polish groups
- R-modules
- real/complex Banach spaces $\longrightarrow$ vector spaces with a Banach cover
- Banach spaces over a non-Archimedean valued field
- Fréchet spaces

Application: injective and projective objects

As an application of this explicit description, we obtain the:

Theorem (Bergfalk, L., Moraschini, Sarti, 2023)

Characterization of injective and projective objects in the left heart of:

- *locally compact abelian Polish groups*
- *totally disconnected locally compact abelian Polish groups*
- *abelian Lie groups*
- *topological torsion locally compact abelian Polish groups*
- *locally compact abelian Polish topological p-groups*
- *locally compact abelian Polish groups with finite ranks*

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Definable refinements of algebraic invariants

Virtually all group invariants from algebraic topology can be refined and seen as invariants taking values in the category of groups with Polish cover

Advantages of the definable versions:

1. finer invariants (distinguish more spaces, more powerful invariants)
2. rigid invariants (fewer automorphisms, better grasp on the dynamics)
3. richer invariants (e.g., one can study their Borel class and Borel rank)

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Finer invariants

Theorem (Bergfalk, L., Panagiotopoulos, 2018–2022)

The following invariants admit definable refinements:

- *Steenrod homology of compact spaces*
- K*-homology of compact spaces and of C*-algebras*
- *Čech cohomology of locally compact spaces*

Furthermore:

1. *definable Steenrod homology* $H_*(-)$ *is a complete invariant for solenoids (inverse limits of tori)*
2. *definable* K*-homology is a complete invariant for solenoids*
3. *definable Čech cohomology* $H^*(-)$ *is a complete invariant for mapping telescopes of tori or spheres*

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Definable homological algebra

The homological invariants

$$\mathrm{Hom}(A, B)$$

$$\mathrm{Ext}(A, B)$$

for countable groups A and B can be seen as groups with a Polish cover.

Theorem (Bergfalk, L., Panagiotopouos, 2019)

Definable $\mathrm{Ext}(-, \mathbb{Z})$ *is a fully faithful functor from finite-rank torsion-free abelian groups with no free summands to groups with a Polish cover*

The definable homological invariant $\mathrm{Ext}(-, \mathbb{Z})$ *is a complete invariant for finite-rank torsion-free abelian groups with no nonzero free summands.*

This does not hold for the purely algebraic Ext.

Ulam stability

Theorem

All Borel-definable automorphisms of $\mathrm{Ext}(\mathbb{Z}[1/p], \mathbb{Z})$ *are given by*

$$\mathrm{Aut}(\mathbb{Z}[1/p]) \curvearrowright \mathrm{Ext}(\mathbb{Z}[1/p], \mathbb{Z})$$

Thus there exist $\aleph_0$ *Borel-definable automorphisms of* $\mathrm{Ext}(\mathbb{Z}[1/p], \mathbb{Z})$

In contrast, there exist $2^{2^{\aleph_0}}$ *automorphisms of* $\mathrm{Ext}(\mathbb{Z}[1/p], \mathbb{Z})$

Let $\mathbb{Q}_p$ be the p-adic numbers (seen as additive locally profinite group)

We have that the action

$$\mathrm{Aut}(\mathbb{Z}[1/p]) \curvearrowright \mathrm{Ext}(\mathbb{Z}[1/p], \mathbb{Z})$$

can be identified with the action by multiplication

$$\mathbb{Z}[1/p]^{\times} \curvearrowright \mathbb{Q}_p/\mathbb{Z}[1/p]$$

Applications to topology: solenoids

A solenoid is simply an inverse limit of copies of $\mathbb{T}$

A concrete geometric realization in $\mathbb{R}^3$ of a solenoid can be obtained as intersection of a sequence of nested solid tori

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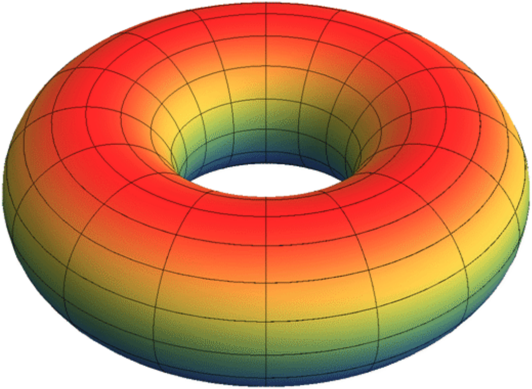

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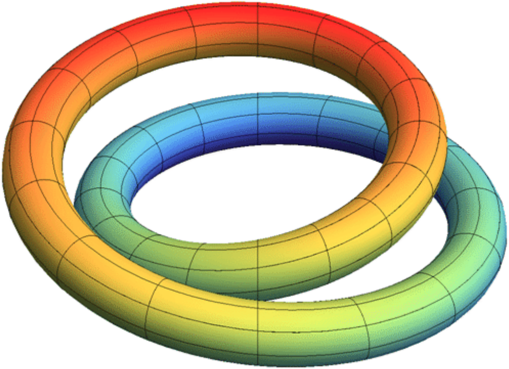

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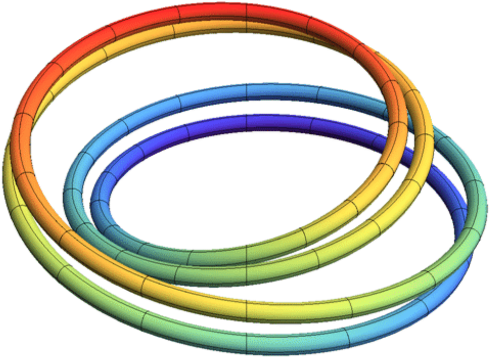

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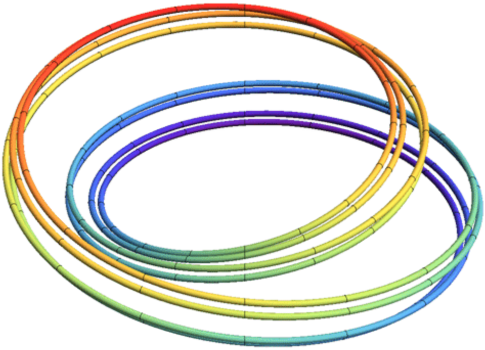

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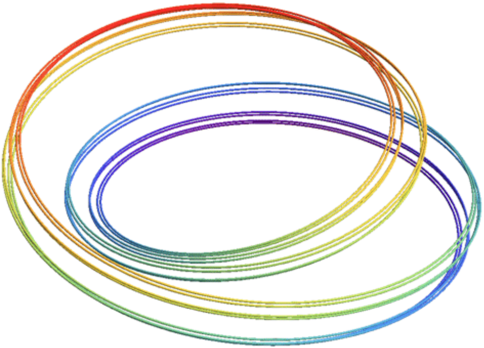

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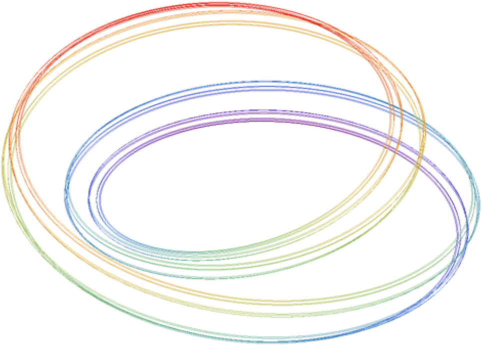

Applications to topology: solenoid complements

We denote by S^3 the one-point compactification of $\mathbb{R}^3$

Let $X_p \subseteq S^3$ be a geometric realization of the p-adic solenoid

Let $[S^3 \setminus X_p, S^2]$ be the space of homotopy classes of maps $S^3 \setminus X_p \to S^2$

Lemma

We have a Borel-definable bijection

$$[S^3 \setminus X_p, S^2] \cong \mathrm{Ext}(\mathbb{Z}[1/p], \mathbb{Z})$$

Theorem (Bergfalk, L., Panagiotopoulos, 2020)

The action

$$[S^3 \setminus X_p, S^2] \curvearrowleft \mathcal{E}(S^3 \setminus X_p)$$

corresponds to the canonical action

$$\mathbb{Z}[1/p]^\times \curvearrowright \mathrm{Ext}(\mathbb{Z}[1/p], \mathbb{Z})$$

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Subobjects

Let $G = \hat{G}/N$ be a group with a Polish cover.

A subgroup with a Polish cover H of G is of the form

$$H = \hat{H}/N$$

for some Polishable subgroup $\hat{H}$ of $\hat{G}$ containing N.

Such a subgroup H of G has a Borel class and a Borel rank.

These are by definition the Borel class and the Borel rank of $\hat{H}$ in $\hat{G}$.

Solecki subgroups

Theorem (L., 2021, building on Solecki 1999 and Farah–Solecki 2006)

Let G be a group with a Polish cover, and let α be a countable ordinal.

There exists a smallest $\boldsymbol{\Pi}_{1+\alpha+1}$ subgroup with a Polish cover $s_\alpha(G)$ of G.

Remark

We have that $s_0(G)$ is the closure of $\{0\}$.

Solecki subgroups for Ext of torsion groups

Theorem (Eilenberg–MacLane, 1942)

The closure of $\{0\}$ *in* $\mathrm{Ext}\,(A, B)$ *is equal to the first Ulm subgroup*

Theorem (L., 2021)

For every countable ordinal α*, and torsion groups A and B,*

$$s_\alpha\,(\mathrm{Ext}(A, B))$$

is equal to the $(1+\alpha)$*-th Ulm subgroup*

$$u_{1+\alpha}(\mathrm{Ext}(A, B))$$

Corollary

For torsion groups A, B, $\{0\}$ *can have arbitrarily high rank in* $\mathrm{Ext}(A, B)$
The problem of classifying extensions can have arbitrarily high complexity.

Solecki subgroups for Ext of torsion-free groups

Theorem (Eilenberg–MacLane, 1942)

The closure of $\{0\}$ in $\mathrm{Ext}\,(A, B)$ parametrizes pure extensions

Theorem (L., 2022)

For a torsion-free A

$$s_1\left(\mathrm{Ext}(A, B)\right)$$

is the subgroup corresponding to finite-rank-pure extensions

Theorem (L., 2023)

For torsion-free A, B,

$$s_1\left(\mathrm{Ext}(A, B)\right) = \{0\}$$

whenever at least one between A, B is sum of finite-rank groups

Complexity and the UCT for cohomology

Corollary

Classification by (integral) cohomology is equivalent to reducibility to E_0^ω

Corollary (Hopf classification)

Let X *be a countable CW-complex with* $H^k(X) = 0$ *for* $k > n$*.*

Then homotopy of maps $X \to S^n$ *is Borel-reducible to* E_0^ω*.*

Complexity and the UCT for K-homology

Corollary

Let X be a compact metrizable space.

Equivalence of extensions of $C(X)$ by $\mathcal{K}$ is Borel reducible to E_0^{ω}

Theorem

Let A be a "well-behaved" separable C-algebra.*

Equivalence of extensions of A by $\mathcal{K}$ is Borel reducible to E_0^{ω}.

Further directions

Project

Caracterization of all Solecki subgroups of Ext *in the torsion-free case*

Project

Construction of torsion-free groups with Ext *of arbitrarily high rank*

Project

Hierarchies of phantom maps corresponding to Solecki subgroups

Project

Generalization to l.c. modules over rings of integers of number fields

Project

Definable invariants in coarse geometry and geometric group theory

Open positions

Open call for 12 PhD Fellowships at the University of Bologna (UNIBO)

1 PhD Fellowship in Logic, funded by ERC grant DAT

`https://www.unibo.it/en/teaching/phd/2023-2024/mathematics`

1 Post-doctoral Fellowship in Logic, funded by ERC grant DAT & UNIBO

`https://bandi.unibo.it/ricerca/assegni-ricerca?id_bando=66430`